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ABSTRACT

A model of first order Fermi acceleration at shock fronts, developed in a previous paper, is extended to consider the angular distribution of the accelerated particles. In the model, particles trapped between a hydromagnetic shock front and a magnetic mirror moving toward it are accelerated by a first order Fermi process. Scattering in pitch angle is required to offset the decrease in pitch angle due to reflection from the moving mirror. The previous paper discussed the general model in the limit that each scattering was through an angle of order unity. Here we consider the limit in which the scattering is weak, although still strong enough to offset the effect of the mirror. A Fokker-Planck differential equation is derived which describes the dependence of the trapped particle distribution on pitch angle, energy and time. It is found that, under conditions of physical interest, the angular distribution quickly relaxes to a characteristic distribution depending only on the mirror ratio B_M / B_O and the variation of the scattering with pitch angle. An approximate energy spectrum of accelerated particles is then readily obtained and it is found that acceleration may be much more efficient than in the strong scattering limit. In particular, whereas formerly a mirror ratio $B_M / B_o \sim 7 - 10$ was required to fit the observed energetic electron spectra near the earth's bow shock, a value of 2 may suffice for weak scattering.

I. INTRODUCTION

In a previous paper (Jokipii, 1965a), referred to here as paper 1, a model of first order Fermi acceleration of charged particles, trapped between a strong shock and an approaching magnetic mirror, was developed in some detail. It was shown that application of the model to the acceleration of electrons at the earth's bow shock led to an understanding of many features of the energetic electron pulses observed beyond the magnetopause (Fan, Gloeckler and Simpson, 1964, 1965; Anderson, Harris and Paoli, 1965; Montgomery, Singer, Conner and Stogsdill, 1965). The model may also be applicable in other astrophysical contexts where shocks are expected.

In paper I, the behavior of the trapped particles was treated under the assumption of strong scattering in pitch angle, and the distribution was averaged over pitch angle. In the present paper, the behavior of the trapped particles is treated, taking into account the pitch angle dependence, by using the Fokker-Planck differential equation. The general differential equation is derived and a solution presented which is valid in the limit of small mirror velocity. It is found that the general results of paper I remain valid, but that acceleration may be much more efficient for a given mirror ratio if the scattering parameter is small. There is, of course, a lower bound on the allowed values of the scattering parameter, since scattering must be sufficient to overcome the decrease in pitch angle due to reflection from the moving mirror. It turns out that whereas, in the limit of strong scattering, the mirror ratio required to fit the observed electron spectra beyond the magnetosphere was $B_M / B_0 \sim 7-10$ (see discussion in paper I), the present analysis suggests that $B_M / B_0 \sim 2$

may actually suffice for certain values of the scattering parameter. It is also determined that, while strong scattering leads to a definite prediction for the energy spectrum, the energy spectrum for weak scattering is determined by the dependence of the scattering parameter on particle energy.

II. THE MODEL

We consider, as in paper I, particles trapped between a magnetic mirror of constant mirror ratio B_M / B_0 and a steady plane hydromagnetic shock front. The magnetic field $\underset{\sim}{B}_0$, into which the shock propagates, is assumed uniform, except for the mirror, and is not parallel to the shock. Without loss of generality, we work entirely in a coordinate system in which the shock is stationary and the plasma flow velocities are parallel or antiparallel to the average magnetic field on either side of the shock (Bayzer and Ericson, 1959). In this frame the mirror is assumed to move toward the shock with uniform velocity, and is a distance $L(t) = L_0 - \beta_0 c\,(t - t_0)$ along $\underset{\sim}{B}_0$ from the shock at time t.

The particles, having rest mass m_0, are reflected from the mirror if their pitch angles $\theta_p = \cos^{-1}\mu = \cos^{-1}|P_{\parallel}/P|$ are outside of the loss cone of the mirror. Here $P_{\parallel}$ is the component of the particle's momentum P parallel to $\underset{\sim}{B}_0$, and the loss cone is defined by $\mu > \mu_c$ where, if $\beta_0^2 << 1$, μ_c satisfies (paper I, equation (2))

$$\frac{1-\mu_c^2}{\left[\mu_c + \beta_0\left(1 + m_0^2 c^2/P^2\right)^{\frac{1}{2}}\right]^2} = \frac{B_0/B_M}{1 - B_0/B_M} . \qquad (1)$$

In most cases of interest the particle velocity $V_p = c\,\beta_p$ is such that

$\beta_P >> \beta_o$. In this case equation (1) reduces to $\mu_c \simeq \sqrt{(1 - B_o/B_M)}$
One can readily demonstrate that each reflection by the moving mirror, of a particle of total energy W and momentum P increases the particle's energy by an amount

$$\Delta W = 2\beta_0 [cP\mu + \beta_0 W] \tag{2}$$

and increases $|P_{\parallel}|$ by

$$\Delta(|P_{\parallel}|) = 2\beta_o [P\mu + \beta_0 \frac{W}{c}] \quad , \tag{3}$$

where we again neglect β_0^2 compared to unity.

Reflection at the shock is more complex and a detailed treatment would require a model for the structure of the collisionless shock, which is not at present available. To illustrate features of the present model, we make the following assumptions on the reflection at the shock. In the reference system chosen the average electric field at the shock is curl free and we assume that a particle does not gain or lose energy upon reflection at the shock. It is essential to the present model, however, that trapped particles are scattered in pitch angle and it is most reasonable that this scattering occur at the shock. Absence of scattering would lead to rapid loss of particles at the loss cone since each reflection at the moving mirror increases μ . Such scattering at the shock is indeed reasonable if the magnetic field at the shock front changes appreciably in a distance

$\lambda \lesssim r_c$, or in a time $\tau \lesssim 1/\Omega_c$, where r_c and Ω_c are, respectively, the cyclotron radius and cyclotron frequency of the particles under consideration. We parametrize this scattering by the r.m.s. scattering angle per reflection $\sigma_\theta = \sqrt{\langle \Delta\theta^2 \rangle}$, where $\langle \Delta\theta^2 \rangle$ is the mean square change in the direction of the momentum due to scattering at the shock. We further assume $\langle \Delta\theta \rangle = 0$. Clearly σ_θ is in general a function of both W and μ. In the absence of knowledge of the shock structure $\sigma_\theta(W, \mu)$ must remain a free parameter in the analysis. Another quantity of interest, the probability of reflection at the shock, should also be considered. In paper I it was concluded that the turbulent magnetic field behind the shock caused the shock to have effectively unit reflectivity. This conclusion has been somewhat strengthened by a recent analysis of the motion of energetic electrons behind the earth's bow shock, where it was found that the turbulent magnetic field is possibly quite effective in preventing motion of the electrons relative to the magnetic field (Jokipii, 1965b). We shall therefore assume the shock to have unit reflectivity, although this is not absolutely essential to the analysis.

In the limit that σ_θ is large, of order unity, the pitch angle distribution of particles reflected at the shock is nearly isotropic over the forward hemisphere and the results of paper I apply. We proceed now to consider the opposite limit, $\sigma_\theta \ll 1$. In this limit we may derive a Fokker-Planck type differential equation for the particle distribution function $n(W, \mu, t)$. Define $n(W, \mu, t)\, dW d\mu \ cm^{-2}$ as the number of particles in the range W to $W + dW$, μ to

$\mu + d\mu$ which are found between the shock and the mirror, per unit area normal to $\underset{\sim}{B}_o$. Let $S(W, \mu, t)\,dW\,d\mu$ cm^{-2}sec^{-1} be the rate that particles are injected into the region between the shock and the mirror. The general equation for $n(W, \mu, t)$ can then be written (Chandrasekhar, 1943)

$$\frac{\partial n}{\partial t} = \lim_{\Delta t \to 0} \left\{ -\frac{\partial}{\partial W}\left(n \frac{\langle \Delta W \rangle}{\Delta t}\right) - \frac{\partial}{\partial \mu}\left(n \frac{\langle \Delta \mu \rangle}{\Delta t}\right) + \frac{1}{2}\frac{\partial^2}{\partial W^2}\left(n \frac{\langle \Delta W^2 \rangle}{\Delta t}\right) + \frac{1}{2}\frac{\partial^2}{\partial \mu^2}\left(n \frac{\langle \Delta \mu^2 \rangle}{\Delta t}\right) + \frac{\partial^2}{\partial W \partial \mu}\left(n \frac{\langle \Delta \mu \Delta W \rangle}{\Delta t}\right)\right\} + S(W, \mu, t) \,, \tag{4}$$

where $\langle \Delta W \rangle$ is the mean change in W over a time Δt, etc., and we have introduced the source function $S(W, \mu, t)$. The problem is to compute the various mean values contained in equation (3). We first note that each trapped particle makes $\frac{dq}{dt} = V_p \mu / 2L(t)$ round trips per unit time, where V_p is the particle velocity. Since we consider $\Delta t \to 0$ we can approximate $\langle \Delta f \rangle / \Delta t$ by $\frac{dq}{dt} \langle \Delta f \rangle / \Delta q$, where $\langle \Delta f \rangle / \Delta q$ is the mean charge in f per reflection from shock or mirror. One can readily determine that, if β_o^2 can be neglected compared to unity and if $\beta_o << V_p/c$,

$$\frac{\langle \Delta W \rangle}{\Delta t} \simeq \frac{\beta_o c \mu^2}{L(t)} W \left(1 - \frac{m_o^2 c^4}{W^2}\right) \tag{5}$$

$$\frac{\langle \Delta W^2 \rangle}{\Delta t} \simeq 0 \,, \tag{6}$$

where terms of order Δt on the right are neglected. Now, $\Delta\mu$ is affected by two competing processes, the scattering at the shock front and the acceleration by the moving mirror. We consider these separately. The change at the mirror is readily obtained from equation (3) and can be written

$$\left(\frac{\langle\Delta\mu\rangle}{\Delta t}\right)_M = \frac{\beta_0 c}{L(t)}\,\mu(1-\mu^2)\sqrt{\left(\frac{W^2-m_0^2c^4}{W^2+m_0^2c^4}\right)} \quad . \tag{7}$$

The scattering in pitch angle has been assumed to be represented by $\sigma_\theta^{\,2}$, with $\langle\Delta\theta\rangle = 0$. If the scattering changes the direction of motion by an angle θ'', and if φ is the azimuthal angle of the new velocity with the old velocity as an axis, simple geometry yields

$$\cos\theta_p' = \cos\theta_p\cos\theta'' + \sin\theta_p\sin\theta''\cos\varphi \ ,$$

where θ_p' and θ_p are, respectively, the new and old pitch angles. In the limit that θ'' is small, we readily obtain to second order in θ''

$$\mu' \simeq \mu\left(1-\tfrac{1}{2}\theta''^2\right) + \theta''\cos\varphi\sqrt{(1-\mu^2)} \quad . \tag{8}$$

It is then readily shown that

$$\left(\frac{\langle\Delta\mu\rangle}{\Delta t}\right)_{scat} = -\frac{c}{4L}\,\mu^2\sigma_\theta^{\,2}\sqrt{\left(1-\frac{m_0^2c^4}{W^2}\right)} \tag{9}$$

$$\left(\frac{\langle\Delta\mu^2\rangle}{\Delta t}\right)_{scat} = \frac{c}{4L}\,\mu(1-\mu^2)\sigma_\theta^{\,2}\sqrt{\left(1-\frac{m_0^2c^4}{W^2}\right)} \tag{10}$$

(This treatment of scattering in pitch angle is similar to that used by Layton (1957) in a slightly different context.) Finally

$$\frac{\langle \Delta W \Delta \mu \rangle}{\Delta t} = 0 \quad . \tag{11}$$

Substituting these into equation (3), one finally obtains

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial W}\left[n\,\frac{\beta_0 c \mu^2}{L(t)} W\left(1-\frac{m_0^2c^4}{W^2}\right)\right]$$
$$-\frac{\partial}{\partial \mu}\left[n\mu(1-\mu^2)\frac{\beta_0 c}{L(t)}\sqrt{\left(\frac{W^2-m_0^2c^4}{W^2+m_0^2c^4}\right)}-\frac{nc}{4L(t)}\mu^2\sigma_\theta^2\sqrt{\left(1-\frac{m_0^2c^4}{W^2}\right)}\right]$$
$$+\frac{1}{2}\frac{\partial^2}{\partial \mu^2}\left[\frac{nc}{4L(t)}\mu(1-\mu^2)\sigma_\theta^2\sqrt{\left(1-\frac{m_0^2c^4}{W^2}\right)}\right] \quad . \tag{12}$$

Equation (12) is to be solved in the region $t \geq t_0$, $W \geq W_0$ and $0 \leq \mu \leq \mu_c$. Particles with $\mu \geq \mu_c \simeq \sqrt{(1-B_0/B_M)}$ will be assumed to be immediately lost to the system, so that we require $n(W, \mu=\mu_c, t) = 0$. The boundary condition at $\mu = 0$ is also easily determined. Any particle scattered through $\mu = 0$ immediately reappears with $\mu \geq 0$ since $\mu = |P_\parallel/P|$. Thus, $\mu = 0$ is to be regarded as a reflecting boundary and we require $\left(\frac{\partial n}{\partial \mu}\right)_{\mu=0} = 0$ at all times. If $n(t=t_0, W, \mu)$ and $n(t, W=W_0, \mu)$ are known, equation (12) can in principle be solved for $n(W, t, \mu)$. It is clear that in general the solution depends in a complicated manner on the as yet unknown dependence of the parameter σ_θ^2 on μ and W. Since the solution to equation (12) is quite difficult to obtain even for quite simple functional forms of $\sigma_\theta^2(W, \mu)$ the general situation is unpromising.

In most cases of interest we are interested in the energy spectrum and angular distribution of particles at energies far above their injection energy. We find below that for this case, in the limit of small β_0, equation (12) simplifies considerably and useful results may be obtained. In particular, it is possible to estimate the energy spectrum and angular distribution for particle energies larger than the injection energy.

III. LIMIT OF SMALL β_0

We now limit consideration to β_0 small enough that the change in μ due to scattering is appreciably greater than that due to reflection from the moving mirror. That is, roughly, we require

$$1 >> \sigma_\theta >> \beta_0 / \beta_p \quad . \tag{13}$$

In this limit, which will be defined more precisely below, we will find that the angular distribution of any injected particles quickly relaxes to a characteristic distribution which depends only on μ_c and $\sigma_\theta(W, \mu)$. The loss through the loss cone is then readily calculated and the energy spectrum of accelerated particles is readily determined by techniques similar to those employed in paper I. Note that it is this limit which is of most physical interest, since if σ_θ were not large enough to counteract the change in pitch angle due to the moving mirror, few particles would be able to gain appreciable energy before being lost at the loss cone. For 30 keV electrons at the earth's bow shock, where $c\beta_0 \sim 10^8$, this corresponds to $\sigma_\theta >> 10^{-2}$.

In order to examine this limit, we first consider the case $\beta_0 = 0$ There then is no change of the energy distribution with time and equation (12), for each value of W, describes the evolution of the pitch angle distribution in time. Consider a group of particles, with energy W and initial pitch angle distribution $F(\mu)$, impulsively injected into the trapping region at $t = 0$. The subsequent evolution of the distribution is governed by

$$\frac{\partial n}{\partial t} = f(W)\left\{\frac{1}{2}\frac{\partial^2}{\partial\mu^2}\left[n\,\sigma_\theta^{\,2}\mu(1-\mu^2)\right] + \frac{\partial}{\partial\mu}\left(n\,\sigma_\theta^{\,2}\mu^2\right)\right. , \qquad (14)$$

where $f(W) = c\,(1 - m_0^2c^4/W^2)^{\frac{1}{2}}/4L$.

We now look for solutions of equation (14) that have the form $g_\lambda(\mu)\exp(-\lambda t)$ and which satisfy the boundary conditions at $\mu = 0$ and $\mu = \mu_c$. In general, λ will depend on W. This will be found to result in a discrete sequence of eigenvalues λ_i and associated eigenfunctions $g_\lambda(\mu)$. The distribution $F(\mu)$ may be expanded in terms of the eigenfunctions and each component decays exponentially with time constant λ_i^{-1}. It is apparent that if the second allowed value of λ is sufficiently larger than the first, after a time $t_2 \simeq 1/\lambda_2$ the pitch angle distribution will be essentially $g_{\lambda_1}(\mu)$ and decay as $\exp(-\lambda_1 t)$. The eigenvalue equation to be solved is

$$\frac{\partial^2}{\partial\mu^2}\left[\sigma_\theta^2 \mu(1-\mu^2) g(\mu)\right] + 2\frac{\partial}{\partial\mu}\left[\sigma_\theta^2 \mu^2 g(\mu)\right] + \frac{2\lambda g(\mu)}{f(w)} = 0 , \tag{15}$$

where $\left.\frac{\partial g}{\partial\mu}\right]_{\mu=0} = g(\mu=\mu_c) = 0$. Before proceeding further, we must discuss the dependence of σ_θ^2 on μ. It is perhaps most plausible that the amount of scattering, per interaction, increases with decreasing μ, perhaps as $1/\mu$. This is because the scattered particle spends an amount of time proportional to $1/\mu$ in the region of scattering. In the present analysis σ_θ^2 will be taken to be proportional to $1/\mu$; the general conclusions, however, are not very sensitive to small variations of this functional dependence.

With $\sigma_\theta^2 = \alpha(w)/\mu$, equation (15) becomes Legendre's differential equation; the general solution may be written

$$g_\lambda(\mu) = a\, P_\nu(\mu) + b\, Q_\nu(\mu) , \tag{16}$$

where $P_\nu(\mu)$ and $Q_\nu(\mu)$ are the Legendre functions and where $\nu(\nu+1) = 2\lambda/\alpha(w) f(w) = \lambda^*$. The ratio of a and b and the allowed values of ν are to be chosen such that the boundary conditions at $\mu = 0$ and $\mu = \mu_c$ are satisfied. That is, we must have

$$a P_\nu(\mu_c) + b Q_\nu(\mu_c) = 0 \tag{17}$$

$$a \left.\frac{dP_\nu}{d\mu}\right]_{\mu=0} + b \left.\frac{dQ_\nu}{d\mu}\right]_{\mu=0} = 0 \quad . \tag{18}$$

Thus,

$$2a \sin(\pi\nu/2) + \pi b_\lambda \cos(\pi\nu/2) \quad , \tag{19}$$

and we seek those values of ν for which

$$2 \sin(\pi\nu/2)\, Q_\nu(\mu_c) + \pi \cos(\pi\nu/2) P_\nu(\mu=\mu_c) = 0 \quad . \tag{20}$$

The allowed values of ν or λ^* are readily obtained from equation (20) for a given value of μ_c . In general this must be done numerically, but since only the lowest values will generally be of interest, this is not much of a burden.

To illustrate the magnitudes involved, we note that expansion around $\mu_c = 0$ yields

$$\mu_c^2 \lambda_1^* \sim 2.5 - 2.2\mu_c^2 \tag{21a}$$

$$\mu_c^2 \lambda_2^* \sim 9 + 8.7\mu_c^2 \quad . \tag{21b}$$

Similarly, one could expand around $\mu_c = 1$ using the asymptotic form of the legendre functions. One finds, for $\mu_c = \frac{1}{2}$, that $\lambda_1^* \sim 8$ and $\lambda_2^* \sim 43$. It is also apparent that, as $\mu_c \to 0$, $\lambda_i^* \to \infty$ for all i and as $\mu_c \to 1$, $\lambda_i^* \to 0$. For the intermediate values of μ_c that are of interest, it appears that λ_2^* is appreciably greater than λ_1^*. Note further that if the dependence of σ_θ^2 on μ were changed to, say, $\sigma_\theta^2 \sim b(w)\mu$, λ_1^* for $\mu_c = \frac{1}{2}$ would only change to ~ 7. This suggests that the above values of λ^* are not very sensitive to changes in the dependence of σ_θ^2 on μ.

Since the second eigenvalue is substantially larger than the first, we proceed to consider the following approximation to estimate the energy spectrum for small β_0. First, since we now have a better estimate on the time required to relax to the characteristic distribution, $g_{\lambda_1}(\mu)$ the inequality in equation (13) may be refined. Thus we may say that if

$$t_2 = \frac{2}{\lambda_2^* \alpha(w) f(w)} << \frac{2L}{v_0} , \tag{22}$$

then a given group of injected particles will quickly relax to the first eigenfunction, and the angular distribution is known. Now consider the resulting energy spectrum. Following the approach of paper I, define

$$h(W,t) = \int_0^{\mu_c} n(W,\mu,t)\, d\mu \quad . \tag{23}$$

The average rate of energy gain is readily obtained in terms of $\langle\mu^2\rangle_1$, the average value of μ^2 over the distribution $g_{\lambda_1}(\mu)$. The expression for particle conservation is then

$$\frac{\partial h}{\partial t} \simeq -\frac{\partial}{\partial W}\left[\frac{c\beta_0\langle\mu^2\rangle_1}{L(t)} W\left(1-\frac{m_0^2c^4}{W^2}\right)h\right] - \frac{1}{2}\lambda_1^*\alpha(W)f(W)\,h \quad , \tag{24}$$

where $S(W,\mu,t) = 0$ since we are considering only energies appreciably greater than the injection energy. Equation (22) simply expresses the fact that loss of particles at a rate $h\lambda_1^*\alpha(W)f(W)/2$ must be balanced by acceleration or a change in the number of trapped particles. If the μ dependence of σ_θ^2 is independent of energy, $\langle\mu^2\rangle_1$ and λ_1^* are constant. The solution to equation (23) is then obtainable in quadratures by the technique used in paper I. We are interested in the situation where $n(W,\mu,t=t_0) = 0$ and a constant source of particles peaked at an energy W_2 is turned on at time t_0. This expresses the condition that initially there are no trapped particles, but that at $t = t_0$ either the source is turned on or the magnetic irregularity becomes connected to the shock front. This boundary condition is of general interest, but is particularly relevant to the problem of acceleration at the earth's bow shock. One finds that the energy spectrum is independent of time except for a constantly increasing cutoff at $W_{max}(t)$. If W_a and W_b are between W_2 and $W_{max}(t)$, then

$$\frac{n(W_a)}{n(W_b)} = \left(\frac{W_b^2 - m_0^2c^4}{W_a^2 - m_0^2c^4}\right)^{(1-1/2\langle\mu^2\rangle_1)} \frac{W_b}{W_a} \exp - \left[\frac{\lambda_1^*}{8\beta_0\langle\mu^2\rangle_1}\int_{W_b}^{W_a}\frac{\alpha(W_1)\,dW_1}{\sqrt{(W_1^2 - m_0^2c^4)}}\right], \tag{25}$$

and $W_{max}(t)$ is given by

$$\frac{W_{max}^2 - 1}{W_2^2 - 1} = \left[\frac{L(t)}{L_0}\right]^{-2\langle\mu^2\rangle_1}. \tag{26}$$

These results correspond to those of paper I except that, while the energy spectrum is uniquely determined in the case of strong scattering, the present spectrum depends on the detailed variation of the scattering parameter with energy. The general results of paper I, except for the shape of the energy spectrum, thus remain valid in the case of weak scattering.

For purposes of illustration, we display the resulting spectrum for two simple functional forms of $\alpha(W)$. The argument used above to obtain the μ dependence of $\sigma_\theta^2 \propto 1/\mu$ suggests that $\alpha(W) \sim K_1/V_p$, where K_1 is constant and $V_p(W)$ is the particle velocity. Substitution in equation (25) yields

$$\frac{n(W_a)}{n(W_b)} = \frac{W_b}{W_a}\left(\frac{W_a^2 - m_0^2c^4}{W_b^2 - m_0^2c^4}\right)^{-\gamma}, \tag{27a}$$

where

$$\gamma = 1 - \frac{1}{2\langle\mu^2\rangle_1} + \frac{\lambda_1^* K_1}{16\beta_0\langle\mu^2\rangle_1}. \tag{27b}$$

If, however, $\alpha(W) = K_2$ is independent of energy, one finds

$$\frac{n(W_a)}{n(W_b)} = \frac{W_a}{W_b}\left(\frac{W_b^2 - m_0^2c^4}{W_a^2 - m_0^2c^4}\right)^{(1 - 1/2\langle\mu^2\rangle_1)}$$

$$\left(\frac{W_a - \sqrt{(W_a^2 - m_0^2c^4)}}{W_b - \sqrt{(W_b^2 - m_0^2c^4)}}\right)^{-\lambda_1^* K_2/8\beta_0\langle\mu^2\rangle_1} . \qquad (28)$$

The spectrum in equation (28) is the same as that obtained in paper I (equation 15) if $\lambda_1^* K_2 / 8\beta_0 \langle\mu^2\rangle_1$ is replaced by b/a .

It was determined in paper I that a spectrum of the form given by equation (28) agreed very well with that observed for the electron pulses beyond the magnetosphere if $\lambda_1^* K_2 / 8\beta_0 \langle\mu^2\rangle_1 \simeq 15 - 20$ for energies ~ 50 keV. Since the parameters must also satisfy the inequality given by equation (21), we can estimate the value of μ_c (and hence B_M / B_0) required to produce this spectrum. We set $c\beta_0 \sim 10^{-8}$, corresponding to the earth's bow shock, and require the inequality in equation (21) to be satisfied by a factor of 5 - 10. One then finds that $\lambda_1^* \sim 4 - 8$ suffices to produce the spectrum. Thus $\mu_c \sim 0.5 - 0.8$, or $B_M / B_0 \sim 2$ may give a sufficiently small loss rate. A similar conclusion is reached if the spectrum in equation (27) is required to fit the observed electron spectrum.

That a mirror ratio of this surprisingly small magnitude may suffice can perhaps be better understood on the basis of the following crude considerations. If $\mu_c \sim 0.6$, a value of $c\beta_0 \sim 10^{8}$ will bring an average 10 keV electron, having $\mu \sim 0.3$ initially, to the loss cone after about 50 reflections. We require the loss rate due to scattering to be substantially larger; suppose that an average particle is scattered out of the loss cone after 10 reflections. This is equivalent to losing 10% of the particles on each cycle. Noting that, non-relativistically, each reflection increases the parallel velocity by about $2\beta_0 c$, the approximate slope of the energy spectrum is readily computed and found to be consistent with the observed slope. It is apparent that the efficiency of the acceleration is due to the high velocity of the electrons, so that the change in μ at the mirror is small. Acceleration of 10 keV protons is much less efficient. In summary, then, the observed spectrum of magnetosheath electrons can be produced by the Fermi process if $B_M / B_0 \sim 2$. This is as opposed to $B_M / B_0 \sim 7$ required in the limit of strong scattering. It should be emphasized that the amount of scattering in pitch angle must be determined before any definite predictions can be made. Nonetheless, it appears that the model of first order Fermi acceleration may be much more efficient than that expected on the basis of previous calculations.

We also observe that the pitch angle distribution of accelerated particles may be highly anisotropic. In particular, it is found that if $\mu_c \sim \frac{1}{2}$, $\langle \mu^2 \rangle_1 \sim 0.1$, so that the ratio $\langle W_\perp \rangle / \langle W_\parallel \rangle$ may be of the order of 10 for the characteristic distribution.

IV. DISCUSSION

The foregoing calculations have been presented to illustrate the effect of weak scattering in pitch angle on particles undergoing first order Fermi acceleration. This is an extension of the analysis of paper I in which a model of Fermi acceleration at shock fronts was developed in some detail and applied to the earth's bow shock. The basic results of paper I, derived under the assumption of strong scattering in pitch angle, remain valid in the case of weak scattering, except that the energy spectrum and angular distribution of the accelerated particles is in general different. Here strong scattering means that particles are scattered through an angle of order unity in each interaction, whereas weak scattering implies that each individual scattering is much less than unity.

In the latter limit a Fokker-Planck equation may be derived which describes the evolution of the trapped particle distribution as a function of energy, pitch angle, and time. This equation has been solved for small mirror velocity $c\beta_0$, where it is found that the distribution relaxes to a characteristic distribution for which $\langle W_\perp \rangle > \langle W_\parallel \rangle$. Observations of the pitch angle distribution of accelerated particles would help to determine the validity of the present theory.

It was found in paper 1 that this model provides an explanation of many features of the energetic electron pulses observed near the earth's bow shock. For strong scattering, a mirror ratio $B_M/B_0 \sim 7$ was required to give the observed spectrum. We conclude on the basis of the present calculations that $B_M/B_0 \sim 2$ may suffice for certain values of the scattering parameter. Clearly this makes the model much more attractive.

It is also found that the energy spectrum may take on many forms in the present model, depending on the functional dependence of the scattering parameter on energy. Wentzel (1965), who considered a different model of first order Fermi acceleration in solar flares, also concluded that a variety of energy spectra may result from the acceleration.

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